J. Fluid Mech. (1981), *vol.* 107, *pp.* 487–497
Printed in Great Britain

Waves in a rapidly rotating gas

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(Received 18 March 1980 and in revised form 18 July 1980)

The eigenvalue problem for the 'acoustic' modes in an inviscid, perfect gas with a quiescent state of isothermal, uniform rotation in a circular cylinder is solved asymptotically for $A^2 \uparrow \infty$ with $(\gamma-1)A^2 = O(1)$, where A is the peripheral Mach number and γ is the specific heat ratio. The limit $(\gamma-1)A^2 \downarrow 0$ leads to a solution in terms of the confluent hypergeometric function for all A, and the resulting eigenvalue equation is solved explicitly for either $A^2 \lesssim 1$ or $A^2 \gg 1$. Attention is focused on those modes (likely to be of greatest practical importance) for which the peripheral speed of the wave relative to that of the container tends to the sonic speed as $A^2 \uparrow \infty$. Viscosity and heat conduction are significant in an inner domain of low density, wherein the solution is expressed in terms of a generalized hypergeometric function.

1. Introduction

The axial and transverse wave motions of an inviscid, perfect gas relative to a quiescent state of isothermal, uniform rotation in a circular cylinder have been considered by Morton & Shaughnessy (1972), who obtained numerical solutions of the eigenvalue problem and present graphical results for eigenvalues and mode shapes, and by Gans (1974), who considered especially the limit $\gamma \downarrow 1$, where γ is the specific heat ratio.† These wave motions are important in the dynamical analysis of a gas centrifuge. I consider them here on the assumptions that $\gamma - 1 \ll 1$ and (in §§4 and 5) that the rotational speed is sufficiently large to render the density small outside of a thin layer adjacent to the lateral wall of the container. The latter condition, which is realized in current practice, simplifies the inviscid model but renders viscosity and heat conduction significant in an inner domain of low density (where the kinematic viscosity and thermal diffusivity, being inversely proportional to the density, are both large), and both must be incorporated in the mathematical model in order to obtain physically acceptable solutions in this domain (cf. Yanowitch 1967).

† Gans (1974) appears to have been unaware of Morton & Shaughnessy's (1972) paper (I, in turn, had been unaware of Gans's paper, which was brought to my attention by one of the referees of the present paper). Gans allows for a one-parameter (his α) family of quiescent temperature distributions (ranging from isothermal to adiabatic) in his initial formulation; however, he obtains explicit results only for $\gamma \downarrow 1$, in which limit his one-parameter family collapses into a single member (there being no difference between isothermal and adiabatic conditions for $\gamma = 1$). There is very little overlap between our respective treatments of the eigenvalue problem for $\gamma \downarrow 1$. Gans omits the factor $(\lambda^2 - 4A^2)^{-1}$ in his (5.3), the counterpart of (3.6) below, and hence obtains the spurious eigenvalues $\lambda = \pm 2A$ ($\lambda = \pm 2$ in his notation), although he recognizes that $\lambda = -2A$ is suspect. Moreover, he holds his a, the counterpart of κ, (3.3) below, fixed in the limit $A^2 \to \infty$ and thereby excludes the only significant asymptotic limit, $\lambda^2 \sim n^2 + \alpha^2$ as $A^2 \to \infty$ with $n, \kappa = O(1)$. He does not consider the inner and outer limits that are examined here in §§4 and 5.

The ambient pressure and density in an isothermal, rotating gas in a circular cylinder of radius r_0 and length l are given by

$$\hat{p}/p_0 = \hat{\rho}/\rho_0 = e^{-\gamma\xi}, \quad \xi = \tfrac{1}{2}A^2\{1-(r/r_0)^2\}, \tag{1.1a, b}$$

where

$$A = \Omega r_0/c_0, \quad c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}; \tag{1.2a, b}$$

p_0 and ρ_0 are the pressure and density at the cylinder wall ($r = r_0$), c_0 is the sonic velocity (which is independent of r by virtue of isothermality), γ is the specific heat ratio, Ω is the angular velocity, A is the peripheral Mach number, and ξ is a dimensionless coordinate that varies from 0 at the wall to $\frac{1}{2}A^2$ at the axis. The assumption of rapid rotation implies

$$\delta \equiv 1/A^2 = \gamma p_0/(\rho_0\Omega^2 r_0^2) \ll 1, \tag{1.3}$$

which suggests that the disturbances to be considered have a scale height δr_0 and are concentrated in $\xi = O(1)$ (but see below).

The equations that govern small, inviscid perturbations about the state of uniform rotation are formulated in § 2. This formulation, which is equivalent to those of Morton & Shaughnessy (1972) and Gans (1974), culminates in a pair of linear, first-order, homogeneous differential equations that govern the complex amplitudes of the perturbation pressure and radial velocity and contain two dimensionless wavenumbers, α (axial) and n (azimuthal), as parameters and a reduced frequency, λ, as an eigenvalue. The resulting eigenvalue problem admits a three-parameter family of solutions, the third parameter being the number of radial zeros. The parametric domain of greatest practical interest is $\alpha \ll 1$ (modern gas centrifuges are long) and $n = 1$ (the only motions that couple with transverse motions of the centrifuge). The dominant mode (smallest λ, no radial zeros) for prescribed α and n decays monotonically away from the wall and is likely to be the most important for high rotation speeds; however, the higher modes may be significant for disturbances that originate in the outer flow (outside of the wall layer).

The solution in the limit $(\gamma-1)A^2 \downarrow 0$ is expressed in terms of the confluent hypergeometric function in § 3. The resulting eigenvalue problem is solved approximately for all of the eigenvalues for moderate A by approximating the confluent hypergeometric function by a Bessel function. These approximations have only qualitative validity for $|A| \gg 1$, but they do describe the general trends and provide heuristic evidence that all but one (for fixed α and n) of the eigenvalues recede to infinity in the limit $A^2 \uparrow \infty$. An asymptotic approximation to this dominant eigenvalue is obtained through an asymptotic approximation to the confluent hypergeometric function.

Asymptotic approximations for the dominant mode(s) in the limit $A^2 \uparrow \infty$ with $(\gamma-1)A^2 = O(1)$ are developed in § 4. The domain of these approximations is $\xi = O(A)$, rather than $\xi = O(1)$, because the scale height of both the perturbation velocity and the ratio of the perturbation pressure to the ambient pressure is $O(r_0/A)$; however, the scale height of both the kinetic energy and the perturbation pressure is $O(r_0/A^2)$, as anticipated above.

The effects of viscosity and heat conduction in the inner domain are included in § 5, and an analytical solution is obtained in terms of generalized hypergeometric functions under the joint restrictions $A^2 \gg 1$ and $\gamma - 1 \ll 1$ after approximating the Prandtl number by unity. This solution matches the first outer approximation of § 4.

It should be emphasized that, although a proper recognition of this inner domain is essential for the completion of the solution in the outer domain, viscous dissipation is concentrated in the boundary layers at $r = r_0$ and $z = 0, l$ (the Ekman layers), which are not considered here.

2. Inviscid formulation

The continuity, Euler (radial, azimuthal and axial components), and energy equations that govern small perturbations about the equilibrium state of uniform rotation in the inviscid flow are

$$D\rho + r^{-1}(r\hat{\rho}u)_r + r^{-1}\hat{\rho}v_\theta + \hat{\rho}w_z = 0, \tag{2.1a}$$

$$\hat{\rho}(Du - 2\Omega v) = -p_r + \Omega^2 r\rho, \tag{2.1b}$$

$$\hat{\rho}(Dv + 2\Omega u) = -r^{-1}p_\theta, \tag{2.1c}$$

$$\hat{\rho}Dw = -p_z \tag{2.1d}$$

and

$$D(p - c_0^2\rho) - (\gamma - 1)\,\hat{p}_r u = 0, \tag{2.1e}$$

where

$$D = \partial_t + \Omega\,\partial_\theta, \tag{2.2}$$

subscripts imply partial differentiation, p and ρ are the perturbation pressure and density, u, v, w are the radial, azimuthal and axial components of the velocity, and r, θ, z are cylindrical co-ordinates in the rotating reference frame. The boundary conditions for free oscillations are

$$u = 0 \quad (r = r_0), \quad ru = 0 \quad (r = 0), \quad w = 0 \quad (z = 0, l). \tag{2.3a, b, c}$$

The boundary conditions (2.3c), together with the requirement that the solution be single-valued in θ ($0 \leqslant \theta \leqslant 2\pi$), are satisfied by expanding in Fourier series and positing the component solutions

$$\{p, \rho\} = \{c_0^2 P, R\}\,\hat{\rho}\,e^{i(n\theta - \sigma t)}\cos kz, \tag{2.4a}$$

$$\{u, v\} = c_0\{-i(r_0/r)\,Q, V\}\,e^{i(n\theta-\sigma t)}\cos kz \tag{2.4b}$$

and

$$w = ic_0\,W\,e^{i(n\theta-\sigma t)}\sin kz, \tag{2.4c}$$

where

$$k = m\pi/l \equiv \alpha/r_0, \quad \lambda = (n\Omega - \sigma)\,r_0/c_0, \tag{2.5a, b}$$

m and n are integers, $\hat{\rho}$ is the ambient density, and P, Q, R, V and W are dimensionless functions of r/r_0. Substituting (2.4) into (2.1) and eliminating R, V and W, we obtain

$$\lambda\left\{x\frac{d}{dx} + \frac{2nA}{\lambda} + (\gamma-1)\,A^2x^2\right\}P + \{\lambda^2 - 4A^2 - (\gamma-1)\,A^4x^2\}\,Q = 0 \tag{2.6a}$$

and

$$\{n^2 - (\lambda^2 - \alpha^2)\,x^2\}\,P + \lambda\left\{x\frac{d}{dx} - \frac{2nA}{\lambda} + A^2x^2\right\}Q = 0, \tag{2.6b}$$

where

$$x = r/r_0. \tag{2.7}$$

The boundary conditions (2.3a, b) go over to

$$Q = 0 \quad (x = 1), \quad Q = 0 \quad (x = 0). \tag{2.8a, b}$$

3. The limit $\gamma \downarrow 1$

Letting $(\gamma-1)A^2 \downarrow 0$ with $\lambda^{-1} = O(1)$ [which restriction rules out the rotational modes, for which $\lambda^2 = O((\gamma-1)A^4)$] in (2.6), eliminating Q,† and introducing

$$\eta = \tfrac{1}{2}A^2(r/r_0)^2 \equiv \eta_0(r/r_0)^2, \tag{3.1}$$

we obtain

$$Q = 2(4A^2-\lambda^2)^{-1}\lambda\{\eta(d/d\eta)+(nA/\lambda)\}P \tag{3.2a}$$

and

$$\{\eta(d/d\eta)^2+(1+\eta)(d/d\eta)+\kappa+\tfrac{1}{2}-\tfrac{1}{4}n^2\eta^{-1}\}P = 0, \tag{3.2b}$$

where

$$\kappa = -\tfrac{1}{2}+\frac{nA}{\lambda}+\frac{(\lambda^2-4A^2)(\lambda^2-\alpha^2)}{2A^2\lambda^2}. \tag{3.3}$$

The differential equation (3.2b) has a regular singularity with exponents $\pm\frac{1}{2}n$ at $\eta = 0$, and that solution which satisfies (2.8b) is given by

$$P = P_0(\eta/\eta_0)^{\frac{1}{2}n}\, e^{\eta_0-\eta} M(\eta)/M(\eta_0), \tag{3.4}$$

where

$$M(\eta) \equiv {}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}n-\kappa; 1+n; \eta) \tag{3.5}$$

is Kummer's (confluent hypergeometric) function. Substituting (3.4) into (3.2a) and invoking (2.8a), we obtain the eigenvalue equation

$$(\lambda^2-4A^2)^{-1}[\eta\{M'(\eta)/M(\eta)\}+n(A\lambda^{-1}+\tfrac{1}{2})-\eta] = 0 \quad (\eta = \eta_0 = \tfrac{1}{2}A^2). \tag{3.6}$$

It is worth noting that (3.4) reduces (2.4a) to

$$p/p_0 = P_0(r/r_0)^n\{M(\eta)/M(\eta_0)\}\, e^{i(n\theta-\sigma t)}\cos kz \quad (\gamma = 1). \tag{3.7}$$

The eigenvalue equation (3.6) is analytically intractable without further approximation. If $n, \alpha = O(1)$ ($n = 1, \alpha \ll 1$ is the domain of greatest practical interest) the available approximations depend on the relative values of κ and η_0. We consider first the régime $|\kappa| \gg \eta_0$,‡ in which [S (3.8.3); the prefix S designates an equation or section in Slater (1960)]

$$M(\eta) = n!\, e^{\frac{1}{2}\eta}(\tfrac{1}{2}z)^{-n}\{J_n(z)+\tfrac{1}{8}(n+1)\kappa^{-1}\eta J_{n+2}(z)+\ldots\}, \quad z = 2(\kappa\eta)^{\frac{1}{2}}. \tag{3.8a, b}$$

Retaining only the dominant term in (3.8a), substituting into (3.6), and imposing the restriction $\lambda^2 \neq 4A^2$, we obtain

$$zJ_n'(z)/J_n(z) \equiv F_n(z^2) = -2nA\lambda^{-1}+\tfrac{1}{2}A^2, \tag{3.9a}$$

where

$$z^2 = 2\kappa A^2 = (\lambda^2-4A^2)(\lambda^2-\alpha^2)\lambda^{-2}+2nA^3\lambda^{-1}-A^2. \tag{3.9b}$$

† It is simpler to eliminate P and solve for Q if $n = 0$, in which case an exact, confluent-hypergeometric-function solution is obtained without the restriction $(\gamma-1)A^2 \ll 1$; see appendix A.

‡ The inequality $|\kappa| \gg \eta_0 \equiv \frac{1}{2}A^2$ holds for $A \to 0$ and fails for $A^2 \to \infty$; however, (3.9a) rests essentially on the modelling of $M(\eta)$ by the dominant term in the expansion (3.8a) and has a much wider domain of utility than $A^2 \ll 1$; cf. Tricomi's approximations to the zeros of $M(\eta)$ (S§6.1.3).

The eigenvalue equation (3.9a) is similar to one obtained by Lamb (1932, § 210) in connection with surface waves in a rotating basin. There is a discrete, infinite sequence of eigenvalues (of z) that tend to the zeros of $J_n'(z)$, namely $j_{n,m}'$ ($m = 1, 2, \ldots$), as $A \to 0$. Graphical comparison of $F_n(z^2)$ and the right-hand side of (3.9a), *qua* function of z^2, reveals that the eigenvalues of z^2, say $z_{n,m}^2$, increase/decrease with increasing A^2 if $A\lambda \gtrless 0$ [although the approximation (3.9a) presumably deteriorates for large A^2]. It also follows from this graphical comparison that $z_{n,m}^2 > 0$ for all $A\lambda < 0$ if $m > 1$ but that $z_{n,1}^2 < 0$ for sufficiently large values of A^2 if and only if $A\lambda < 0$; for example, (3.9) implies $z_{1,1}^2 < 0$ if $A\lambda < 0$ and $|A| > 2-\sqrt{2}$ ($z_{1,1}^2 = 0$ at $|A| = 2-\sqrt{2}$ and $\lambda = -\sqrt{2}\,\mathrm{sgn}\,A$).

The eigenvalues of λ may be expanded about $A = 0$ by expanding $F(z^2)$ about $z^2 = j_{n,m}'^2 \equiv z_0^2$, using the differential equation

$$2y(dF/dy)+F^2+y-n^2 = 0, \quad y = z^2. \tag{3.10}$$

The second approximation is [it is convenient to assume $\lambda > 0$, there being no loss of generality by virtue of the invariance of the eigenvalue problem under the transformation $(-\lambda, A) \to (\lambda, -A)$]

$$\lambda^2 = \alpha^2+z_0^2+4nAz_0^2(z_0^2-n^2)^{-1}(z_0^2+\alpha^2)^{-\frac{1}{2}}+O(A^2) \quad (z_0 \equiv j_{n,m}'). \tag{3.11}$$

The third approximation with $\alpha = 0$ is

$$\lambda^2 = z_0^2+4nAz_0(z_0^2-n^2)^{-1}+A^2\{5-z_0^2(z_0^2-n^2)^{-1}-8n^2z_0^2(z_0^2-n^2)^{-3}\}+O(A^3) \quad (\alpha = 0). \tag{3.12}$$

This last approximation is consistent with Morton & Shaughnessy's (3.5), which holds only for $n = 0$, after expanding their result in powers of A. It also yields

$$\lambda^2 = 3{\cdot}39+3{\cdot}08A+1{\cdot}59A^2 \quad (n = 1, \quad m = 1, \quad \alpha = 0) \tag{3.13a}$$

and

$$\lambda^2 = 28{\cdot}42+0{\cdot}78A+3{\cdot}95A^2 \quad (n = 1, \quad m = 2, \quad \alpha = 0). \tag{3.13b}$$

A fair approximation for all but the dominant mode for any n and α and for moderate values of A is given by

$$\lambda^2 = j_{n,\mathrm{m}}'^2+\alpha^2+4(n/j_{n,m}')A+4A^2 \quad (A^4 \ll 4j_{n,\mathrm{m}}'^2, m > 1). \tag{3.14}$$

For example, the values of λ given by (3.14) for $\alpha = 0$, $n = 1$ and $m = 2$ are in error by 2·5% for $A = 3$ and 9% for $A = 4$.

The preceding approximations fail for sufficiently large A but appear to be adequate for at least rough estimates of the eigenvalues in $|A| \lesssim m+1$ with the important exception of the dominant mode ($m = 1$) in that parametric domain in which $z_{n,1}^2 < 0$ ($A \lesssim -0{\cdot}6$ for $n = 1$). It does not appear to be possible to obtain a satisfactory approximation to λ from (3.9) in this domain except in the neighbourhood of $z_{1,1} = 0$ (see above); accordingly, we seek an asymptotic approximation as $A \downarrow -\infty$.

A suitable asymptotic approximation to $\log M$ for $\eta \gg |\kappa| \gg 1$ is given by [S(4.5.5) for $\kappa > 0$, S(4.5.21) for $\kappa < 0$]

$$\log M(\eta) \sim K+\tfrac{1}{2}\eta\left\{1+\left(1-\frac{4\kappa}{\eta}\right)^{\frac{1}{2}}\right\}+(\kappa-\tfrac{1}{2}n-\tfrac{1}{2})\log\eta-\tfrac{1}{4}\log\left(1-\frac{4\kappa}{\eta}\right)$$
$$+2\kappa\log\left|1-\left(1-\frac{4\kappa}{\eta}\right)^{\frac{1}{2}}\right|+O(\eta^{-1}) \quad (\eta \gg \kappa \gg 1), \tag{3.15}$$

where K is a constant. Substituting the derivative of (3.15) into (3.6) and letting $A \downarrow -\infty$ with $\lambda = O(1)$, we obtain (after a straightforward but lengthy reduction)

$$\lambda^2 = n^2+\alpha^2+O(A^{-2}) \quad (A \downarrow -\infty). \tag{3.16}$$

This result also appears [after invoking the appropriate counterpart of (3.15)] to hold for $A \uparrow \infty$, but the numerical results of Morton & Shaughnessy (1972, figure 4) and the approximation (3.9) suggest that (3.16) is valid only for $A \downarrow -\infty$. The cause of the spurious result for $A \uparrow \infty$ is not clear at this time.

The differential equation obtained by eliminating Q between (2.6*a*, *b*) and retaining $O(\gamma-1)$ as $\gamma \downarrow 1$ also may be transformed to the confluent hypergeometric equation. The resulting solution is

$$P = P_0(\eta/\eta_0)^{\frac{1}{2}n} \exp\{bc(\eta_0-\eta)\} M(c\eta)/M(c\eta_0), \tag{3.17}$$

where

$$b = 1+(\gamma-1)(\alpha/\lambda)^2, \quad c = 1+(\gamma-1)\{1-2(\alpha/\lambda)^2+2A^2(\lambda^2-4A^2)^{-1}\}, \tag{3.18a, b}$$

M is given by (3.5) with

$$\kappa = -\tfrac{1}{2}+\frac{nA}{\lambda}\left\{1+2(\gamma-1)\left(\frac{\alpha^2}{\lambda^2}-1\right)\right\}+\frac{(\lambda^2-4A^2)(\lambda^2-\alpha^2)}{2A^2\lambda^2}\left\{1+(\gamma-1)\left(\frac{2\alpha^2}{\lambda^2}-1\right)\right\} + \tfrac{1}{2}(\gamma-1)(nA/\lambda)^2, \tag{3.19}$$

and $O\{(\gamma-1)^2\}$ error terms are implicit in each of (3.18*a*, *b*) and (3.19). The counterpart of (3.6) is

$$\{\lambda^2-4A^2-(\gamma-1)A^4\}^{-1}[c\eta_0 M'(c\eta_0)+\{n(A\lambda^{-1}+\tfrac{1}{2})+(\gamma-1-bc)\eta_0\}M(c\eta_0)] = 0. \tag{3.20}$$

4. The limit $\delta \downarrow 0$ (outer approximation)

It appears from the results of the preceding section that, at least for $\gamma = 1$, all of the eigenvalues recede to infinity as $A \to \pm\infty$ except that for the dominant mode (one for each of $n = 0, 1, \ldots$) as $A \downarrow -\infty$, which is given by (3.16). We consider here the asymptotic approximation to this mode in the limit $\delta \equiv 1/A^2 \downarrow 0$ with $\xi = O(1)$ (but see below), where ξ is given by (1.1*b*). Since $\gamma-1$ typically is comparable with δ in magnitude, we let $\gamma-1 = O(\delta)$, but this parametric regime does not comprise the rotational modes (see Morton & Shaughnessy, § 4).

Introducing ξ in place of x in (2.6), we obtain

$$\lambda\{(d/d\xi)-\gamma_1\delta-2\omega(1-2\delta\xi)^{-1}\}P+\{\gamma_1+(4-\delta\lambda^2)(1-2\delta\xi)^{-1}\}Q = 0 \tag{4.1a}$$

and

$$\delta\{\lambda^2-\alpha^2-n^2(1-2\delta\xi)^{-1}\}P+\lambda\{(d/d\xi)-1+2\omega(1-2\delta\xi)^{-1}\}Q = 0, \tag{4.1b}$$

where

$$\gamma_1 = (\gamma-1)A^2, \quad \omega = n(A\lambda)^{-1}. \tag{4.2a, b}$$

It is evident that (4.1) and (2.8*a*) admit the limiting solution

$$\lambda^2 = n^2+\alpha^2, \quad P = P_0 e^{2\omega\xi}, \quad Q = 0 \quad (\delta \to 0). \tag{4.3a, b, c}$$

The requirement that P remain bounded as $\xi \uparrow \infty$, which replaces the boundary condition (2.8*b*), is satisfied by (4.3*b*) if and only if $A/\lambda < 0$. Note that (4.3*a*) and (2.4) then imply that the peripheral speed of the disturbance relative to that of the container is equal to the sonic speed c_0.

The approximation (4.3) provides the basis for an outer expansion of the solution in $\xi = O(\omega^{-1}) = O(\delta^{-\frac{1}{2}})$, from which it appears that the natural scale for P, which measures the perturbation pressure *relative to* the ambient pressure, is $\delta^{\frac{1}{2}}r_0$ rather than δr_0; nevertheless, it proves more efficient to regard both ω and ξ as if they were $O(1)$ and to pose the expansion in the form

$$\lambda^2 = \lambda_0^2 + \delta\Lambda_1 + \delta^2\Lambda_2 + \dots, \tag{4.4a}$$

$$P = P_0 e^{2\omega\xi}\{1 + \delta P_1(\xi) + \delta^2 P_2(\xi) + \dots\} \tag{4.4b}$$

and

$$Q = P_0 e^{2\omega\xi}\{\delta^2 Q_2(\xi) + \dots\}, \tag{4.4c}$$

where $P_0 \equiv P(0)$ is a constant. This last condition, together with (2.8*a*), implies

$$P_n(0) = Q_n(0) = 0 \quad (n > 0). \tag{4.5}$$

Substituting (4.4) into (4.1), equating powers of δ with $\omega = O(1)$, and integrating the resulting equations for P_1, P_2 and Q_2 subject to (4.5), we obtain

$$\lambda_0^2 = n^2 + \alpha^2, \quad \Lambda_1 = 2n^2(1-4\omega)^{-1}, \quad \Lambda_2 = 8n^2(1-4\omega)^{-3} + 2\gamma_1 n^2(1-4\omega)^{-2}, \tag{4.6a, b, c}$$

$$P_1 = \gamma_1\xi + 2\omega\xi^2, \quad P_2 = \tfrac{1}{2}P_1^2 + \tfrac{1}{2}(\gamma_1+4)(\Lambda_1/\lambda_0^2)\,\xi^2 + \tfrac{8}{3}\omega\xi^3 \tag{4.7a, b}$$

and

$$Q_2 = -(\Lambda_1/\lambda_0)\,\xi. \tag{4.8}$$

The resulting second approximation to λ (assumed positive) is

$$\lambda = (n^2+\alpha^2)^{\frac{1}{2}} + n^2 A^{-1}\{A(n^2+\alpha^2)^{\frac{1}{2}} - 4n\}^{-1} + O(A^{-4}). \tag{4.9}$$

The first approximation to the dimensionless perturbation pressure may be placed in the form [see (2.4*a*), where the factor $\exp(-\gamma\xi)$ is introduced]

$$e^{-\gamma\xi}P/P_0 = e^{-\xi}(1-2\delta\xi)^{-nA/\lambda}\{1+O(\delta^2)\} \equiv P^{(1)}. \tag{4.10}$$

The corresponding second approximation is

$$P^{(2)} = e^{-\xi}(1-2\delta\xi)^{-nA/\lambda}\left\{1 + \left(\frac{\gamma-1}{2} + \frac{2}{A^2}\right)\left(\frac{\lambda^2-n^2-\alpha^2}{n^2+\alpha^2}\right)\xi^2 + O(\delta^3)\right\}. \tag{4.11}$$

The approximations (4.9)–(4.11) are compared with the numerical results of Morton & Shaughnessy (1972) for the dominant mode (their 'mode 1') for $n = 1, \alpha = 0, \gamma = 1{\cdot}06$ and $-5 \leqslant A \leqslant -1$ in tables 1 and 2. [Morton & Shaughnessy's numerical results are for $\gamma = 1{\cdot}06$, not $\gamma = 1{\cdot}4$ as stated in their paper (Morton, private communication).]

$-A$	$\lambda(4.9)$	$\lambda(\mathrm{M\&S})$
1	1·200	1·270
2	1·083	1·102
3	1·048	1·055
4	1·031	1·035
5	1·022	1·024

TABLE 1. λ for $n = 1$, $\alpha = 0$ (dominant mode) and $\gamma = 1{\cdot}06$, as given by the approximation (4.9) and the numerical results of Morton & Shaughnessy (1972).

	A								
	-1			-3			-5		
r/r_0	$P^{(1)}$	$P^{(2)}$	$e^{\gamma\xi}P/P_0$ (M & S)	$P^{(1)}$	$P^{(2)}$	$e^{\gamma\xi}P/P_0$ (M & S)	$P^{(1)}$	$P^{(2)}$	$e^{\gamma\xi}P/P_0$ (M & S)
1	1	1	1	1	1	1	1	1	1
0·9	0·763	0·769	0·774	0·233	0·237	0·239	0·0332	0·0342	0·0344
0·8	0·576	0·592	0·600	0·055	0·059	0·061	0·0013	0·0014	0·0015
0·7	0·428	0·451	0·465	0·013	0·015	0·017	$5{\cdot}2\times10^{-5}$	$6{\cdot}3\times10^{-5}$	$7{\cdot}7\times10^{-5}$
0·6	0·311	0·337	0·358	0·003	0·004	0·005	$2{\cdot}3\times10^{-6}$	$3{\cdot}0\times10^{-6}$	$4{\cdot}9\times10^{-6}$

TABLE 2. The dimensionless perturbation pressure, as given by (4.10), (4.11), and the numerical results of Morton & Shaughnessy (1972).

5. Interior diffusion

Both heat conduction and viscosity may be significant in the domain of low density. Retaining only the dominant effects of diffusion in the limit $\delta \to 0$, we obtain†

$$D\rho + r^{-1}(r\hat{\rho}u)_r + r^{-1}\hat{\rho}v_\theta + \hat{\rho}w_z = 0, \tag{5.1a}$$

$$\hat{\rho}(Du - 2\Omega v) = -p_r + \Omega^2 r\rho, \tag{5.1b}$$

$$\hat{\rho}(Dv + 2\Omega u) = -r^{-1}p_\theta + \mu v_{rr}, \tag{5.1c}$$

$$\hat{\rho}Dw = -p_z + \mu w_{rr}, \tag{5.1d}$$

and

$$D(p - c_0^2\rho) - (\gamma-1)\,\hat{p}_r u = (\mu/\mathsf{P})\{(\gamma p - c_0^2\rho)/\hat{\rho}\}_{rr}, \tag{5.1e}$$

in place of (2.1), where μ is the viscosity and P is the Prandtl number. Substituting (2.4) into (5.1) and imposing the approximation $\mathsf{P} = 1$, we obtain [cf. (4.1)]

$$[\lambda\{\mathscr{D} - \gamma_1\delta - 2\omega(1-2\delta\xi)^{-1}\} + i\mathscr{E}\mathscr{D}]\,P + \{\gamma_1 + (4-\delta\lambda^2)(1-2\delta\xi)^{-1} - i\delta\lambda\mathscr{E}\}\,Q = 0 \tag{5.2a}$$

and

$$\delta\{\lambda^2 - \alpha^2 - n^2(1-2\delta\xi)^{-1} + i\gamma\lambda\mathscr{E}\}\,P + [\lambda\{\mathscr{D} - 1 + 2\omega(1-2\delta\xi)^{-1}\} + i\mathscr{E}(\mathscr{D}-\gamma)]\,Q = 0, \tag{5.2b}$$

where γ_1 and ω are defined by (4.2),

$$\mathscr{D} = d/d\xi, \quad \mathscr{E} = \epsilon e^{\gamma\xi}\mathscr{D}^2, \quad \epsilon = \mu/(\delta^2\rho_0 c_0 r_0). \tag{5.3a, b, c}$$

† It can be shown that viscous diffusion is negligible in the radial equation of motion for $\delta \ll 1$. Axial diffusion is significant in the Ekman layers at $z = 0, l$, but these layers are of only secondary importance in the present context.

We now assume that $\gamma - 1 = O(\delta)$ and $\lambda^2 = \alpha^2 + n^2 + O(\delta)$, as in §4, and let $\delta \downarrow 0$ with $\delta \ln \epsilon = o(1)$ [so that $2\delta\xi \ll 1$ for $\epsilon e^{\gamma\xi} = O(1)$], $P = O(1)$ and $Q = O(\delta)$, in which limit (5.2) reduces to

$$\{\lambda(\mathscr{D} - 2\omega) + i\mathscr{E}\mathscr{D}\} P = 0 \tag{5.4a}$$

and

$$\{\lambda(\mathscr{D} - 1 + 2\omega) + i\mathscr{E}(\mathscr{D} - 1)\} Q = -i\delta\lambda\mathscr{E}P \tag{5.4b}$$

or, on introducing [the present problem is *not* invariant under the transformation $(\lambda, -A) \to (-\lambda, A)$, so we do not assume $\lambda > 0$]

$$\zeta = (|\lambda|/\epsilon)\, e^{-\xi + \frac{1}{2}i\pi \operatorname{sgn} \lambda}, \quad \Delta = \zeta(d/d\zeta), \quad a = 2\omega, \tag{5.5a, b, c}$$

$$\{\Delta^3 - \zeta(\Delta + a)\} P = 0 \tag{5.6a}$$

and

$$\{\Delta^2(\Delta + 1) - \zeta(\Delta + 1 - a)\} Q = \delta\lambda\Delta^2 P. \tag{5.6b}$$

The differential equation (5.6a) has a regular singularity at $\zeta = 0$, with exponents $\rho_1 = \rho_2 = \rho_3 = 0$, and an irregular singularity at $\zeta = \infty$. The method of Frobenius (Ince 1944, § 16.3) yields the three linearly independent solutions

$$F_1(\zeta) = {}_1F_2(a; 1, 1; \zeta) = \sum_0^\infty c_n \zeta^n, \tag{5.7a}$$

$$F_2(\zeta) = -\sum_0^\infty (c_n \ln \zeta + c_n') \zeta^n, \tag{5.7b}$$

and

$$F_3(\zeta) = \sum_0^\infty (c_n \log^2 \zeta + 2c_n' \log \zeta + c_n'') \zeta^n, \tag{5.7c}$$

where

$$c_n = \frac{\Gamma(a+n)}{\Gamma(a)\,\Gamma^3(n+1)}, \quad c_n' = c_n\{\psi(a+n) - 3\psi(n+1)\}, \tag{5.8a, b}$$

$$c_n'' = c_n\{\psi(a+n) - 3\psi(n+1)\}^2 + c_n\{\psi'(a+n) - 3\psi'(n+1)\}, \tag{5.8c}$$

and $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$. The generalized hypergeometric function ${}_1F_2(a; 1, 1; \zeta)$ diverges like $\zeta^{\frac{1}{2}a - \frac{2}{3}} \exp(2\zeta^{\frac{1}{2}})$ as $\zeta \to \infty$ in $|\arg \zeta| < \pi$ (see appendix B) and hence must be excluded from P in consequence of the requirement that $Q = 0$ at $\xi = 0$. The function F_3 also must be excluded from P in consequence of the requirement that $Q \to 0$ as $\zeta \to 0$ ($\xi \uparrow \infty$); its inclusion would imply $Q = O(\log^2 \zeta)$ as $\zeta \to 0$. This leaves only $F_2(\zeta)$, for which $P = O(\log \zeta)$ and $Q = O(\zeta \log \zeta)$ as $\zeta \to 0$ [but note that the perturbation pressure, as given by (2.4a), is $O(\zeta \log \zeta)$]. Letting

$$P = AF_2(\zeta), \quad P_0 = AF_2(\epsilon^{-1}|\lambda|\, e^{\frac{1}{2}i\pi \operatorname{sgn} \lambda}), \tag{5.9a, b}$$

and invoking (appendix B)

$$F_2 \sim \{\Gamma(a)/\Gamma(1-a)\} \zeta^{-a} \quad (\zeta \to \infty, \quad a \neq 0, -1, -2, \ldots), \tag{5.10}$$

we obtain

$$A = P_0\{\Gamma(1-a)/\Gamma(a)\} (|\lambda|/\epsilon)^a e^{\frac{1}{2}i\pi a \operatorname{sgn} \lambda}. \tag{5.11}$$

The preceding development does not take advantage of the fact that $a = 2\omega$ is small in the present context. Letting $a \to 0$ in (5.7*b*), we obtain (after some manipulation)

$$F_2(\zeta) = a^{-1} - 2C - \ln\zeta + a\{S(\zeta) - \tfrac{1}{6}\pi^2\} + O(a^2), \tag{5.12}$$

where

$$S(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n(n!)^2}\left\{2\psi(n+1) + \frac{1}{n} - \ln\zeta\right\} \tag{5.13a}$$

$$= 4\int_0^{2\surd\zeta} \{K_0(t) + \ln(\tfrac{1}{2}t) + C\}\frac{dt}{t}, \tag{5.13b}$$

$C = 0{\cdot}577\ldots$ is Euler's constant, and K_0 is a modified Bessel function. Substituting (5.12) into (5.9*b*) or, alternatively, letting $a \to 0$ in (5.11), we obtain

$$A = aP_0[1 + a\{\ln(|\lambda|/\epsilon) + 2C + \tfrac{1}{2}i\pi\,\mathrm{sgn}\,\lambda\} + O(a^2)]. \tag{5.14}$$

This work was supported by the U.S. Department of Energy under Contract no. DE-AC05-760R01779 with the University of Virginia and was carried out by the writer in his capacity as a member of the Gas Centrifuge Theoretical Consultants Group. I am indebted to J. B. Morton and H. G. Wood for fruitful discussions.

Appendix A. Axisymmetric modes

Setting $n = \omega = 0$ in (2.6), introducing η from (3.1), and eliminating P, we obtain

$$\{(d/d\eta)^2 + \gamma(d/d\eta) + (\gamma-1)(\alpha/\lambda)^2 + (2-\tfrac{1}{2}\delta\lambda^2)(\alpha^2\lambda^{-2}-1)\eta^{-1}\}Q = 0. \tag{A 1}$$

The required solution then is given by (cf. Morton & Shaughnessy 1972)

$$Q = Ce^{-\frac{1}{2}(\gamma\eta+\zeta)}\zeta M(1-\kappa, 2, \zeta), \tag{A 2}$$

where

$$\zeta = \{\gamma^2 - 4(\gamma-1)(\alpha/\lambda)^2\}^{\frac{1}{2}}\eta, \tag{A 3}$$

$$\kappa = \{\gamma^2 - 4(\gamma-1)(\alpha/\lambda)^2\}^{-\frac{1}{2}}(2A^2\lambda^2)^{-1}(\lambda^2-4A^2)(\lambda^2-\alpha^2), \tag{A 4}$$

and C is a constant. The corresponding eigenvalue equation is

$$M(1-\kappa, 2, \zeta_0) = 0, \tag{A 5}$$

where ζ_0 is given by (A 3) with $\eta = \frac{1}{2}A^2$ therein.

Letting $\zeta_0 \uparrow \infty$ ($A \uparrow \infty$) and invoking S (4.1.6), we obtain

$$M(1-\kappa, 2, \zeta_0) \sim \{\Gamma(1-\kappa)\}^{-1}\zeta_0^{-\kappa-1}e^{\zeta_0}\{1 + O(\zeta_0^{-1})\} + O(\zeta_0^{\kappa-1}), \tag{A 6}$$

from which it follows that the roots of (A 5) are asymptotic to the poles of $\Gamma(1-\kappa)$ or, equivalently,

$$\kappa = m \quad (m = 1, 2, 3, \ldots). \tag{A 7}$$

Note that $\kappa = 0$ is not included in this sequence, but that $\lambda = \alpha$, which implies $\kappa = 0$, is an admissible eigenvalue for which $Q \equiv 0$ (see first paragraph in § 4).

Combining (A 4) and (A 7) and letting $A^2 \uparrow \infty$ with $\gamma - 1 = \gamma_1/A^2$, we obtain

$$\lambda^2 = \frac{2\alpha^2}{m+2} + O\left(\frac{1}{A^2}\right), \quad 2(m+2)A^2 + \left(\frac{m}{m+2}\right)\alpha^2 + 2m\gamma_1 + O\left(\frac{1}{A^2}\right). \tag{A 8a, b}$$

Appendix B. Asymptotic approximations to $F_1(\zeta)$ and $F_2(\zeta)$

The asymptotic behaviour of any function $F(\zeta)$ as $\zeta \to \infty$ is determined by the behaviour of its Laplace transform, $f(s) = \mathscr{L}F$, as $s \to 0$ if, as in the present case, $f(s)$ is singular there. Invoking S (3.2.39), where the prefix S signifies an equation in Slater (1960), we obtain†

$$\mathscr{L}F_1(\zeta) \equiv \int_0^\infty e^{-s\zeta}F_1(\zeta)\,d\zeta = s^{-1}\,{}_1F_1(a;1;s^{-1}) \quad (\mathscr{R}s > 0) \tag{B 1}$$

for the generalized hypergeometric function $F_1(\zeta)$, (5.7*a*). Invoking S (4.1.7),

$$\Gamma(a)\,s^{-1}\,{}_1F_1(a;1;s^{-1}) = s^{-a}e^{1/s}\{1+O(|s|)\} \quad (s \to 0, \quad \mathscr{R}s > 0), \tag{B 2}$$

inverting (B 1), and letting $\zeta \to \infty$, we obtain

$$\Gamma(a)\,F_1(\zeta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\zeta^{\frac{1}{2}a-\frac{3}{4}}\exp(2\zeta^{\frac{1}{2}})\{1+O(\zeta^{-1})\} \quad (\zeta \to \infty) \tag{B 3}$$

for all a. [The inverse transform of (B 2) exists for all ζ if and only if $\mathscr{R}a > 0$, but this restriction is unnecessary for the asymptotic limit $\zeta \to \infty$. The result (B 3) is a special case of Wright's (1935) results for the generalized hypergeometric function. The present derivation is more economical (at least for $\zeta > 0$) than the conventional derivation from a Barnes-contour-integral representation; it also motivates the following derivation.]

The Laplace transform of $F_2(\zeta)$, obtained by term-by-term transformation of (5.7*b*), is given by

$$\mathscr{L}F_2(\zeta) = \Gamma(a)\,s^{-1}U(a;1;s^{-1}) \quad (\mathscr{R}s > 0), \tag{B 4}$$

where U, defined by S (1.5.23, 24), is a second solution of the confluent hypergeometric equation. Invoking S (4.1.11),

$$U(a;1;s^{-1}) \sim s^a\{1+O(|s|)\} \quad (s \to 0, \quad |\arg s| < \tfrac{3}{2}\pi), \tag{B 5}$$

we obtain

$$F_2(\zeta) \sim \{\Gamma(a)/\Gamma(1-a)\}\,\zeta^{-a}\{1+O(|\zeta|^{-1})\} \quad (\zeta \to \infty, \quad |\arg \zeta| < \tfrac{3}{2}\pi). \tag{B 6}$$

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† Slater gives $\mathscr{R}s > 1$ for the validity of S (3.2.39), but it is evident from her preceding result that $\mathscr{R}s > 0$ is sufficient.